

Rotations and Angular Momenta

Finite Versus Infinitesimal Rotations

Consider a vector

$$\mathbf{V} = (V_x \ V_y \ V_z)^T ,$$

after a rotation

$$\begin{pmatrix} V'_x \\ V'_y \\ V'_z \end{pmatrix} = R \begin{pmatrix} V_x \\ V_y \\ V_z \end{pmatrix}$$

with

$$R^T R = R R^T = 1 ,$$

leading to a property

$$\mathbf{V}^T \mathbf{V} = \mathbf{V}^T R^T R \mathbf{V} ,$$
$$V_x'^2 + V_y'^2 + V_z'^2 = V_x^2 + V_y^2 + V_z^2 .$$

Define a rotation operator about the z-axis by angle ϕ ,

$$R_z(\phi) = \begin{pmatrix} \cos \phi & -\sin \phi & 0 \\ \sin \phi & \cos \phi & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

We are particularly interested in an infinitesimal form of R_z :

$$R_z(\epsilon) = \begin{pmatrix} 1 - \frac{\epsilon^2}{2} & -\epsilon & 0 \\ \epsilon & 1 - \frac{\epsilon^2}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix} , \quad \epsilon \rightarrow 0 .$$

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Likewise, we have

$$R_x(\epsilon) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 - \frac{\epsilon^2}{2} & -\epsilon \\ 0 & \epsilon & 1 - \frac{\epsilon^2}{2} \end{pmatrix},$$

and

$$R_y(\epsilon) = \begin{pmatrix} 1 - \frac{\epsilon^2}{2} & 0 & \epsilon \\ 0 & 1 & 0 \\ -\epsilon & 0 & 1 - \frac{\epsilon^2}{2} \end{pmatrix}.$$

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Elementary matrix manipulations lead to

$$R_x R_y = \begin{pmatrix} 1 - \frac{\epsilon^2}{2} & 0 & \epsilon \\ \epsilon^2 & 1 - \frac{\epsilon^2}{2} & -\epsilon \\ -\epsilon & \epsilon & 1 - \epsilon^2 \end{pmatrix}$$

$$R_y R_x = \begin{pmatrix} 1 - \frac{\epsilon^2}{2} & \epsilon^2 & \epsilon \\ 0 & 1 - \frac{\epsilon^2}{2} & -\epsilon \\ -\epsilon & \epsilon & 1 - \epsilon^2 \end{pmatrix}$$

$$R_x R_y - R_y R_x = \begin{pmatrix} 0 & -\epsilon^2 & 0 \\ \epsilon^2 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = R_z(\epsilon^2) - 1,$$

where all terms of order higher than ϵ^2 have been ignored.

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Infinitesimal Rotations in Quantum Mechanics

Given a rotation operation characterized by a orthogonal 3×3 matrix R , associate an operator $\mathcal{D}(R)$ in the appropriate ket space such that

$$|\alpha\rangle_R = \mathcal{D}(R)|\alpha\rangle.$$

- For describing a spin-1/2, system with no other degrees of freedom, $\mathcal{D}(R)$ is a 2×2 matrix;
- for a spin-1 system, $\mathcal{D}(R)$ is a 3×3 matrix.

The appropriate infinitesimal operators could be written as

$$\hat{U}(\epsilon) = 1 - i\hat{G}\epsilon, \quad \hat{G} : \text{Hermitian}$$

We therefore define the angular-momentum operator $\hat{J}_k$ for an infinitesimal rotation around the k th axis by angle $d\phi$ can be obtained by letting

$$\hat{G} \rightarrow \frac{\hat{J}_k}{\hbar}, \quad \epsilon \rightarrow d\phi$$

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